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Optimal exploitation of renewable resources under uncertainty and the extinction of species*

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Summary. We consider an optimally managed renewable resource with stochastic non-concave growth function. We characterize the conditions under which the optimal policy leads to global extinction, global conservation and the existence of a safe standard of conservation. Our conditions are specified in terms of the economic and ecological primitives of the model: the biological growth function, the welfare function, the distribution of shocks and the discount rate. Our results indicate that, unlike deterministic models, extinction and conservation in stochastic models are not determined by a simple comparison of the growth rate and the discount rate; the welfare function plays an important role.

Keywords and Phrases: Renewable resources, Extinction, Biological species, Safe standard of conservation, Optimal resource management, Stochastic dynamic programming.

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1 Introduction

Extinction of biological species is an important ecological concern of the current age. Extinction is likely whenever a renewable resource is harvested persistently at a rate exceeding the level required to sustain its current stock. The economics

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of extinction relates the depletion of resources to economic incentives that affect harvesting. Traditionally, economists have related overexploitation of resources to failures of markets and property rights. However, even if such failures are corrected and society "manages" its resources *optimally*, the nature of intertemporal trade-offs between current and future welfare that a society is willing to make can lead to eventual extinction. It is, therefore, important to understand how intertemporal preferences of society and the biological growth of resources interact to determine the possibility of extinction and conservation for an optimally managed resource.

One important factor here is the sensitivity of biological growth to random environmental fluctuations and the fact that persistent adverse environmental shocks can severely deplete resource stocks. Such environmental uncertainty also affects the incentive to harvest a resource. The nature of optimal exploitation and its effects on the dynamics of biological populations when the growth process of a specie is subject to random environmental shocks are not very well understood. In this paper we undertake a systematic study of this issue and identify conditions under which the following four scenarios occur: (i) the resource becomes extinct (with probability one) starting from all positive initial stocks; (ii) the resource is conserved (with probability one) from all positive initial stocks; (iii) there is a positive resource stock (called a safe standard of conservation) such that, starting from all higher stocks, the resource is conserved (with probability one); (iv) there is no such safe standard of conservation.

Beginning with [4], much of the analysis of the economics of optimal extinction and conservation of renewable resources has been carried out in *deterministic* models. The conventional wisdom from this literature suggests that stocks of an optimally managed resource ought to be bounded away from zero as long as the resource has an intrinsic growth rate[1] that exceeds the rate at which society discounts the future; on the other hand, extinction is optimal if the resource is less productive than the discount rate.[2] However, when the natural growth of the resource is stochastic, comparing productivity of the resource to the discount rate is no longer sufficient to characterize the possibility of extinction. Indeed, optimal stocks may be arbitrarily close to zero no matter how productive the resource.[3]

One of the salient features of the natural growth of many species is that the productivity or biological growth rate is low from small stocks, but it increases as the stock becomes larger, though eventually the growth rate diminishes as the environmental carrying capacity is approached. Therefore, the biological growth or production function in a model of optimal renewable resource management is typically non-concave (such as S-shaped). Our model of optimal resource management is, in fact, identical to a one-sector stochastic optimal growth model with

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[1] The intrinsic growth rate refers to the net productivity of the resource at zero.

[2] If the intrinsic growth rate of the resource is less than the discount rate, then extinction is optimal from small stocks. If the resource is globally less productive than the discount rate, global extinction is optimal. See, among others, [11–13, 7] and [6].

[3] In an optimal stochastic growth model, it was shown in [17] that optimal stocks may not be bounded away from zero even though the production function has infinite slope at zero (that is, the intrinsic growth rate is infinite).

non-concave production function.[4] However, while economic growth models focus on the existence, uniqueness and stability of a non-trivial invariant distribution for capital stocks, our focus is on the phenomenon of extinction; that is, whether the stocks approach zero over time and the probability with which this event occurs.[5]

We explore a model of optimal resource management by using the methods of stochastic dynamic programming. The return function in the optimization exercise (the net welfare function) depends on the harvest of the resource. The biological production function and the optimal investment policy determine the transition function governing the stochastic evolution of the resource stock. We determine conditions on the primitives of the optimization problem, under which global extinction, global conservation and the existence (and non-existence) of a safe standard of conservation will arise.

The existing literature on the problem of characterizing extinction and non-extinction in terms of verifiable properties of the primitives of the dynamic optimization problem (intertemporal preferences and the biological growth or production function), is rather small.[6] In models of optimal stochastic growth, the possibility of extinction is ruled out by assuming that the slope of the production function is infinite at zero and that the worst realization of the random shock occurs with strictly positive probability.[7] An assumption of infinite marginal product at zero is not well suited to our purpose because the rate of natural growth for most biological species is rather small when the stock depletes to a level close enough to zero.[8] Indeed, in contrast to the stochastic growth literature, we wish to understand the phenomenon of extinction and not rule it out by assumption. A recent paper on stochastic growth [8] shows that if the marginal product at zero is finite, every feasible path (including, therefore, any optimal path) converges to zero almost surely provided the random shocks are "sufficiently volatile".

In this paper, we allow the production function to have finite slope at zero and the probability distribution of the random shock is assumed to be non-atomic with

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4 The literature on stochastic optimal growth [3] typically assumes that the production function is concave. For an analysis of the problem of stochastic optimal growth in a framework that allows for a non-concave production function, see [14].

5 From the perspective of economic growth theory, our analysis is relevant to the question of existence and nature of poverty traps in a non-convex economy (even when it is on its first-best path). From a methodological standpoint, it is worth noting that in establishing global stability of invariant distributions, models of economic growth impose fairly strong conditions to ensure that the capital stocks are bounded away from zero. The conditions for avoidance of extinction in our analysis are significantly weaker and suggest that convergence results in stochastic growth models may be obtained for a wider class of production functions.

6 There is a significant literature on characterization of extinction and non-extinction in terms of the transition law for a given Markov process (rather than the primitives of an economic model that generates such transition law). For the special case of multiplicative shock with a smooth density function whose support is the entire positive real line (so that from any current stock one may reach any interval of stocks, however high or low, with strictly positive probability), [18] contains conditions on the transition function under which the stochastic process converges globally to a degenerate distribution at zero (in the norm topology) as well as conditions under which it converges globally to a unique distribution that assigns zero probability mass at zero.

7 See, among others, [3, 16] and [14].

8 See, for example, [5], and the references cited there.

bounded support. We show that the net welfare from harvesting plays an important role in the conditions for ruling out extinction. In general, our conditions are much tighter than in the existing stochastic growth literature.

In the literature on renewable resource management under uncertainty,[9] there is no general analysis of conditions for extinction that can be verified from information about the natural growth of the resource and the net welfare from harvesting. [20] provides sufficient conditions for conservation in a model where an (s,S) investment policy is optimal. When there are no fixed costs, these conditions assume that welfare is linear in consumption and that the resource growth function is strictly *concave*. Specialized models of a similar kind (with specific parametric form) are analyzed in [2] and [9].[10] The linearity in consumption of the welfare function in these models implies that the conditions for conservation are solely determined by the productivity of the resource relative to the discount rate. As our analysis will establish, this does not hold when the net welfare function is non-linear. A more general analysis of the conditions for conservation in a model where the utility function depends on both consumption as well as resource stock is contained in [19]; the condition for conservation provided in our model can be viewed as a special case of that analysis when the marginal utility from resource stock is zero. However, there is no analysis of conditions for non-existence of safe standard and extinction in that paper.

Section 2 outlines the model and assumptions. Section 3 contains preliminary results about the value function and the optimal policy as well as formal definitions of concepts related to extinction and conservation. Section 4 contains the main results of the paper on the conditions for conservation and extinction under optimal management of the resource. All proofs are collected in Section 5.

2 The model

In this section, we outline a model where a renewable resource is harvested over time according to the optimal dynamic decisions of a social planner (or monopoly owner). The planner chooses a sequence of resource consumption (or investment) levels in order to maximize the expected discounted sum of (one-period) social welfare over an infinite horizon given a stochastic production function which summarizes the biological growth possibilities and a known distribution of environmental disturbances.

Time is discrete and is indexed by $t = 0, 1, 2, ...$ The initial stock of the resource $y_0 > 0$ is given. Let $Y = \mathbb{R}_+$ be the state space for resource stocks. At each date $t \geq 0$, the current resource stock $y_t \in Y$ is observed and a harvest or consumption level, c_t, is chosen. The remaining stock represents resource investment or escapement, $x_t = y_t - c_t$. The feasible set for consumption and investment is denoted by $\Gamma(y) = \{(x, c) | 0 \leq c, 0 \leq x, c + x \leq y\}$.

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[9] See, [10] and [5].

[10] [1] analyzes possibility of immediate extinction in a framework where eventual extinction occurs almost surely even if the resource is not harvested.

There are random environmental shocks to the production (stock-recruitment) process of the renewable resource. Let $\{r_t\}$ be an independent and identically distributed random process defined on the probability space $(\Omega, \mathcal{F}, P)$, where the marginal distribution is denoted by μ, and the support of this distribution is given by the interval $I = [a, b]$ with $0 < a < b < \infty$.[11]

The biological growth of the resource is governed by a production function, $f : Y \times I \to Y$, that determines the resource stock next period (gross output) as a function of current investment in the stock and the environmental shock such that $y_{t+1} = f(x_t, r_{t+1})$. The resource production or growth function is assumed to satisfy the following:

(T.1) For all r, $f(x, r)$ is strictly increasing in x; for all x, $f(x, r)$ is non-decreasing in r.

(T.2) For all r, $f(0, r) = 0$.

(T.3) $f(x, r)$ is continuous in (x, r) on $Y \times I$. For each $r \in [a, b], f(x, r)$ is differentiable in x on $\mathbb{R}_{++}$ and, further, $f'(x, r)$ is continuous on $\mathbb{R}_{++} \times I$.

(T.4) There exists $\bar{x} > 0$ such that $f(x, b) < x$ for all $x \geq \bar{x}$; $y_0 \in (0, \bar{x}]$.

Assumptions (T.1)–(T.3) are standard monotonicity and smoothness restrictions on production. Assumption (T.4) is a bounded growth restriction typically associated with a natural carrying capacity for the ecosystem beyond which the resource stock cannot grow.

The lower bound on the intrinsic growth rate (that is, the marginal product at zero investment) is given by the lower right derivative of f, which is denoted by $D_+ f(0, r) = \operatorname{liminf}_{x \downarrow 0} f'(x, r)$. Define $\nu = \inf_{r \in I}[D_+ f(0, r)]$ to be the lower bound on the intrinsic growth rate over all possible realizations of the random shock. We assume

(T.5) $\nu > 0$.

Assumption (T.5) ensures that the marginal product is bounded away from zero. It allows for cases of critical depensation where for each r, $f(x, r) < x$ for all $x > 0$ close to zero, so that extinction is inevitable from small stocks even if the resource is never harvested. Of course, (T.5) also encompasses cases where for each r, the marginal productivity at zero is greater than one so that the resource can sustain itself from small stocks.

For each $r \in [a, b]$, let

$$S(r) = \left\{ \hat{x} \geq 0 : \left[\frac{f(\hat{x}, r)}{\hat{x}}\right] \geq \left[\frac{f(x, r)}{x}\right], \forall x \geq 0 \right\}, \text{if } \liminf_{x \downarrow 0} \left[\frac{f(x, r)}{x}\right] < \infty,$$
$$= \{0\}, \text{ otherwise.}$$

[11] More formally, let Ω be the space of all infinite sequences $(\omega_1, \omega_2, \dots)$ where $\omega_t \in [a, b]$ for $t \in \mathbb{N}$. Denote by $\mathcal{B}$ the collection of Borel subsets of $[a, b]$. Let $\mathcal{F}$ be the σ-algebra generated by cylinder sets of the form $\prod_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{B}$ for all $n \in \mathbb{N}$, and $A_n = I$ for all but a finite number of values of n. For each $t \in \mathbb{N}$, denote by $\mathcal{F}_t$ the σ-algebra generated by cylinder sets of the form $\prod_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{B}$ for all $n \in \mathbb{N}$, and $A_n = I$ for all $n \geq t + 1$. Let P be the product measure over $\mathcal{F}$ generated by the probability distribution μ over $[a, b]$. This defines a probability space $(\Omega, \mathcal{F}, P)$. Next, define the projection $r_t(\omega) = \omega_t$ for $t \in \mathbb{N}$. Then $\{r_t\}_1^{\infty}$ is a sequence of independent and identically distributed random variables on $(\Omega, \mathcal{F}, P)$.

Define

$$\hat{x}(r) = \sup\{x : x \in S(r)\}.$$

Thus, $\hat{x}(r)$ is the highest investment among the set of investments that maximize average productivity corresponding to realization r of the random shock. In the special case of a multiplicative shock, $\hat{x}(r)$ is identical for all r. We assume that:

(T.6) For each $r \in [a, b]$, $f(x, r)$ is concave in x on $[\hat{x}(r), \infty)$.

Typically, biological resources are likely to exhibit low "productivity" or growth rate when the biomass is small and productivity is likely to increase as the biomass expands. That is, the production function is likely to be convex at low levels of investment. As the resource exhibits bounded growth it is reasonable to assume (as we do in assumption (T.6)) that eventually diminishing returns must set in.

Define

$$\hat{x} = \sup_{r \in [a,b]} \hat{x}(r).$$

If $\hat{x} > 0$, then the production function is non-concave for at least some r. If the production function is concave for all r, then $\hat{x} = 0$.

In the bio-economics literature, the population growth process is often specified in terms of a (net) growth function that is non-monotonic (e.g., the logistic curve). In our framework, the net biological growth is given by the function $[f(x, r) - x]$ which is non-monotonic in x. Our assumptions allow for non-concavity in net growth (depensation), negative net growth at small stocks (critical depensation) as well as the usual concave "inverse U-shaped" net growth function (compensation).[12]

The existing literature on resource allocation with non-concave production focuses on models where the resource growth function is S-shaped and where the resource can always be sustained from low stocks. The model of resource growth employed here generalizes these two restrictions. First, it allows for the possibility of critical depensation where the resource is incapable of sustaining itself from low stocks. In such cases, the important question is whether economic efficiency implies conservation of the resource from large stocks. Second, the model in this paper considers a broader class of growth functions than those that are S-shaped. Resource growth is allowed to exhibit almost any pattern of increasing and decreasing returns on the interval $[0, \hat{x}(r)]$.

Finally, we impose a technical restriction:

(T.7) For any $x > 0$, $\mu\{r : f(x, r) > f(x, a)\} = 1$.

Assumption (T.7) implies that in any period, the probability that next period's stock is exactly equal to that obtainable under the worst production function is zero. This has the effect of putting zero measure on the worst production function (which is stronger than just putting zero measure on the worst realization of the random shock).

The net social welfare in each period depends on current consumption and is denoted by $u(c)$. This welfare function can incorporate consumer and producer

[12] See, [5].

surplus from resource harvests. In the bio-economics literature, it is typically viewed as the social benefit from the harvesting (through direct consumption, use in other production processes or though international trade) net of the cost of economic inputs used in harvesting the resource (such as labor and capital used in fishing). For a monopolist's dynamic optimization problem, $u(c)$ would denote the current profit from harvest c.

The objective is to maximize the expected discounted sum of social welfare over time, where $\delta \in (0, 1)$ is the discount factor. Let $\overline{\Re} = \Re \cup \{-\infty\}$. The welfare function satisfies the following restrictions:

(U.1) $u : Y \to \overline{\Re}$ is concave on Y; $\lim_{c \downarrow 0} u(c) = u(0)$.

(U.2) u is continuously differentiable on $\mathbb{R}_{++}$.

(U.3) Either (i) there exists a $\xi > 0$ such that $u'(c) > 0$ for all $c \in (0, \xi)$ and $u'(c) < 0$ for all $c > \xi$, or (ii) $u'(c) > 0$ for all $c \in \mathbb{R}_{++}$.

Assumptions (U.1) and (U.2) are standard. Note that we allow the utility of zero consumption to be $-\infty$. Further, we do not assume strict concavity of u so that, in particular, we allow the utility to be linear. Assumption (U.3) is weaker than the typical assumption in stochastic growth models that u is increasing over the domain of c. It implies that welfare is either increasing or unimodal in c; that is, there is a unique strictly positive consumption that maximizes u. This allows for the possibility that marginal harvest costs might exceed marginal benefits at large harvest levels so that excessive consumption might decrease instantaneous welfare. In case $u'(c) > 0$ for all $c \in \mathbb{R}_{++}$, we define $\xi = \infty$.

3 Preliminaries

In this section, we define basic concepts and outline preliminary results on the optimal policy, the stochastic process of optimal stocks, extinction and conservation.

3.1 Optimal policy

The decision-maker in the stochastic environment (outlined in the previous section) can take decisions dependent on the *history* of past states and decisions. To formalize this decision-making process, we start by defining histories.

The *partial history* at date t is given by $h_t = (y_0, x_0, c_0, \ldots, y_{t-1}, x_{t-1}, c_{t-1}, y_t)$. A *policy* π is a sequence $\{\pi_0, \pi_1, \ldots\}$ where π_t is a conditional probability measure such that $\pi_t(\Gamma(y_t)|h_t) = 1$. A policy is *Markovian* if for each t, π_t depends only on y_t. A Markovian policy is *stationary* if π_t is independent of t.

Associated with a policy π and an initial state y is an expected discounted sum of social welfare:

$$V_\pi(y) = E \sum_{t=0}^{\infty} \delta^t u(c_t),$$

where $\{c_t\}$ is generated by π, f in the obvious manner and the expectation is taken with respect to P.

The *value function* $V(y)$ is defined by:

$$V(y) = sup\{V_\pi(y) : \pi \text{ } is \text{ } a \text{ } policy\}.$$

Assumption (T.4) ensures that, given any policy π, we have $V_\pi(y) < \infty$ for all $y > 0$. We *assume* that:

(V.1) There exists a policy π such that $V_\pi(y) > -\infty$ for all $y > 0$.[13]

Thus, the dynamic optimization problem is well defined and the value is finite from any initial state.

A policy, π^*, is *optimal* if $V_{\pi^*}(y) \geq V_\pi(y)$ for all policies π and all y and $V_{\pi^*}(y) = V(y)$. Standard dynamic programming arguments (see, Theorem 5.2 and Theorem 16.2 in [21])[14] imply that there exists an optimal policy that is stationary and the value function satisfies the functional equation:

$$V(y) = \sup_{x \in \Gamma(y)} [u(y-x) + \delta E[V(f(x,r))]]. \tag{3.1}$$

Further, V is non-decreasing[15] and continuous. Let $X(y)$ be the set of maximizers of the expression on the right hand side of (3.1). Then, $X(y)$ is the optimal policy correspondence. Any function $H(y)$ generates a stationary optimal policy if, and only if, $H(y)$ is a measurable selection from $X(y)$. $X(y)$ is an upper-hemicontinuous correspondence that admits a measurable selection. $X(y)$ will be referred to as the (stationary) optimal investment correspondence, while $C(y) = y - X(y)$ will be called the optimal consumption correspondence. The minimum and maximum selections from $X(y)$ are denoted (respectively) by $X_m(y) = min\{x : x \in X(y)\}$, and $X^M(y) = max\{x : x \in X(y)\}$.

It can be shown that the optimal investment correspondence $X(y)$ has certain monotonicity properties. More precisely:

Lemma 1 *$X_m(y)$ and $X^M(y)$ are non-decreasing in y on Y.*

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[13] If $\nu > 1$ or $u(0) > -\infty$, then this always holds. If neither of these conditions hold (a possibility not ruled out by our assumptions), then this can be ensured if the discount factor is smaller than a critical value that depends on u and ν.

[14] Let $S = [0, \overline{x}]$ be the state space and $A = [0, 1]$ be the action space where choosing an action $a \in A$ when the current stock is y implies that current consumption is ay yielding utility $u(ay)$. As u is bounded above on S,we can choose a modified utility function $W(ay) = u(ay) - B$ where B is a constant that is an upper bound of u on S. Then, $W \leq 0$ and therefore, following the remark in the last paragraph of p. 181 in [21], Condition C in that paper is satisfied. The constraint correspondence is constant valued and hence upper hemi continuous. The weak continuity of the transition law follows from the continuity assumption on f (see, Lemma 2 in [14] for a precise proof of this). Using (U.1), one can check that all of the requirements for Condition W in Section 16 of [21] are satisfied. Theorem 16.2 then implies the existence of a stationary optimal policy (see Statement III on p. 186). Theorem 5.2 establishes the functional equation. The optimal policy for the modified problem can be easily shown to be optimal in the original problem and the functional equation for the original problem can be derived from the modified problem.

[15] Even though u is decreasing on (ξ, ∞), it is easy to check that from every current stock, optimal consumption must necessarily lie in $[0, \xi]$ a.s., so that the standard arguments about monotonicity of V applies.

When resource stocks are optimally managed over time using an optimal investment function $H(y)$, a measurable selection from $X(y)$, the transition function for resource stocks is given by:

$$y_t(y,\omega) = f(H(y_{t-1}(y,\omega)),\omega_t) \ for \ t \geq 1, \tag{3.2}$$

and $y_0(y,\omega) = y$.[16]

Note that the optimal investment correspondence $X(y)$ is upper hemi-continuous but it does not necessarily admit a continuous selection; in particular, both $X_m(y)$ and $X^M(y)$ may be discontinuous.

In general, the optimal investment and/or consumption need not lie in the interior of the feasible set. The following lemma outlines a condition which guarantees that optimal investment in the resource is strictly positive from all stocks $y > 0$.

Lemma 2 *Assume that for all* $y > 0$,

$$\delta \left\{ \lim_{c \downarrow 0} u'(c) \right\} E(D_+ f(0,r)) > u'(y). \tag{3.3}$$

Then $X_m(y) > 0$ *for all* $y > 0$.

The condition outlined in this lemma is always satisfied if $u'(c) \to \infty$ as $c \to 0$. It is also satisfied if the marginal utility of consumption is bounded above but the technology is delta-productive in expected terms; that is, if $E(D_+ f(0,r)) > 1/\delta$. The lemma encompasses the standard Inada condition used to guarantee interior investment in classical optimal growth models (see, for example, [3]).

In the classical growth model, the Inada condition also guarantees that optimal consumption is strictly positive, which our condition does not do. In fact, in the stochastic non-convex dynamic optimization framework, guaranteeing that optimal consumption is strictly positive from all initial stocks by a general verifiable restriction on technology and preferences requires us to impose rather strong conditions. If the production function is concave in input, then an assumption such as $u'(c) \to \infty$ as $c \to 0$ is sufficient to ensure that optimal consumption is strictly positive; this also holds if the production function is non-concave but there is no uncertainty. However, it is not known whether this condition suffices in general when the technology is both stochastic and non-convex. For example, in [14], it is assumed that $u(0) = -\infty$ in order to ensure that optimal consumption is always positive. In [18], it is shown that if the random shock is multiplicative with support equal to the positive real line and has a smooth density function, then optimal consumption policy is strictly positive under the restriction that $u'(c) \to \infty$ as $c \to 0$.

To conclude this subsection, we note that the stochastic Ramsey-Euler equation holds in case of an interior optimal policy.

Lemma 3 *Let* $x(y)$ *be a measurable selection from* $X(y)$ *and* $c(y) = y - x(y)$. *If* $c(\widehat{y}) > 0$ *for some* $\widehat{y} > 0$, *then*

$$u'(c(\widehat{y})) \geq \delta E[u'(c(f(x(\widehat{y}),r)))f'(x(\widehat{y}),r)]. \tag{3.4a}$$

[16] $y_t(y,\omega)$ is $\mathcal{F}_t$ measurable for all $t \in \mathbb{N}$.

where $u'(0) = \lim_{c\downarrow 0} u'(c), f'(0,r) = \liminf_{x\downarrow 0} f'(x,r)$. *If* $c(\widehat{y}) \in (0,\widehat{y})$ *for some* $\widehat{y} > 0$ *and, further,* $c(y) \in (0,y), \forall y \in [f(x(\widehat{y}),a), f(x(\widehat{y}),b)]$ *then:*

$$u'(c(\widehat{y})) = \delta E[u'(c(f(x(\widehat{y}),r)))f'(x(\widehat{y}),r)]. \tag{3.4b}$$

3.2 Concepts of extinction and conservation

We begin with the notion of a *safe standard of conservation* which occupies a central role in the renewable resource literature. In the deterministic literature, a level of resource stock is said to be a *safe standard of conservation* if optimal paths from all initial stocks lying above the standard are bounded below by the standard. In other words, if we know that the current resource stock lies in the region above the standard, we know it will always be there (even though optimal paths may converge to zero from stocks below the standard).

A natural way to extend this concept to the stochastic case is to require that the safe standard be an *almost sure* lower bound for optimal processes of resource stocks, if the initial state lies in the region above the standard. Formally, a stock $y^* > 0$ is said to be a *safe standard of conservation* if :

$$P\left\{\omega \in \Omega : \liminf_{t\geq 0}\ y_t(y,\omega) \geq y^*\right\} = 1, \tag{3.5}$$

for all $y > y^*$. Note that the definition allows the optimal path from y^* itself to be arbitrarily close to zero with positive probability. The potential discontinuity in the optimal policy function implies that the transition function for the optimal stocks in (3.2) may have a jump discontinuity at the safe standard and though the right hand limit of the function at y^* may be large enough to guarantee that the stocks never fall below y^* from any stock $y > y^*$, the same need not be true if the current stock is equal to y^*.

A situation where there are stocks arbitrarily close to zero that are safe standards of conservation is defined as *global conservation.* Under this definition of global conservation, the resource is bounded away from zero independent of the initial stock (provided it is greater than zero). More formally, global conservation requires that for every $\epsilon > 0$, there exists $y^* \in (0,\epsilon)$ such that (3.5) holds. Note that global conservation implies[17]

$$P\{\omega \in \Omega : \liminf_{t\geq 0}\ y_t(y,\omega) > 0\} = 1, \forall y > 0. \tag{3.6}$$

Next, we outline the concept of extinction. Extinction needs to be defined in a way so as to encompass the event that the resource stock is reduced to zero in finite

[17] It is possible to think of a weaker notion of global conservation where

$$P\left\{\omega \in \Omega : \limsup_{t\geq 0}\ y_t(y,\omega) > 0\right\} = 1, \forall y > 0.$$

We do not use this concept as it is extremely difficult to derive verifiable conditions on the primitives of our optimization model that ensures this but not (3.6).

time as well as the event that the stocks, while never being actually reduced to zero, become arbitrarily small over time. *Extinction* is said to occur from an initial stock $y > 0$ if:

$$P\left\{\omega \in \Omega : \lim_{t\to\infty} y_t(y,\omega) = 0\right\} = 1. \tag{3.7}$$

Global extinction is said to occur if extinction occurs from all initial stocks $y > 0$.

Under global extinction, there is no safe standard of conservation. However, a weaker condition than (3.7) under which optimal stocks approach zero infinitely often with positive probability is also sufficient to rule out the existence of a safe standard of conservation.

4 Conditions for extinction and conservation

In this section, we outline verifiable conditions on the technology and intertemporal preferences under which the stochastic process of resource stocks generated by an optimal investment policy is characterized by conservation or extinction. As mentioned earlier, the optimal policy is not necessarily unique in our framework. The conditions we outline in this section ensure conservation or extinction of stocks when harvesting follows *any* stationary optimal policy (i.e., that generated by any measurable selection from the optimal investment correspondence $X(y)$). In particular, we establish conditions that lead to (i) the existence of a safe standard, (ii) global conservation; (iii) non-existence of safe standard of conservation and (iv) global extinction for any any measurable selection from the optimal

4.1 Safe standard of conservation

First, we examine the conditions under which there is a safe standard of conservation. For this purpose, we analyze the stochastic process of optimal resource stocks when the optimal investment function selected from $X(y)$ is the lower bound $X_m(y)$. For any initial stock $y > 0$, consider the sequence of optimal stocks $\{y_t(y)\}$ defined by $y_0(y,\omega) = y$ and:

$$y_t(y,\omega) = f(X_m(y_{t-1}(y,\omega),\omega_t)) \; for \; t \geq 1. \tag{4.1}$$

In order to establish the existence of a safe standard, we will establish conditions which ensure that:

$$f(X_m(\widehat{y}), a) \geq \widehat{y} \; for \; some \; \widehat{y} > 0. \tag{4.2}$$

Using induction, it is easy to see that (4.2) implies that for $y \geq \widehat{y}, t \geq 1$,

$$y_t(y,\omega) = f(X_m(y_{t-1}(y,\omega),\omega_t)) \geq f(X_m(\widehat{y}), a) \geq \widehat{y},$$

with probability one.

Before we go into aspects of the problem that involve intertemporal trade-offs, there is one class of readily identifiable situations in which there is always a safe standard of conservation. If the marginal utility from consumption is negative when investment falls below the level needed to replenish the stock to its current level under the worst productivity shock, then investment by even a myopic agent will be sufficient to at least replenish the stock. Since an optimizing agent never consumes more than a myopic one, this ensures a safe standard.

Proposition 1 *Suppose that $u'(f(x,a)-x) \leq 0$ for some $x > 0$, then the stock $f(x,a)$ is a safe standard of conservation.*

Next, we consider situations where the hypothesis of Proposition 1 does not hold; that is, where marginal welfare from consuming an amount which allows the stock to be replenished to its current level even under the worst shock is positive. This is always true if u is a strictly increasing function. To obtain a tight condition for conservation it is necessary to overcome the technical difficulties caused by the non-convexity in the feasible set for the dynamic optimization problem when the production function is not concave. Our methodology is to first consider the convexified resource allocation problem obtained by taking the convex hull of the production possibility set for each r. Recall the definition of input level $\hat{x}$ in Section 2; $f(x,r)$ is concave in x for all $x \geq \hat{x}$. We derive a condition that ensures a safe standard of conservation (for this modified optimization problem) which lies above $\hat{x}$. This implies that the optimal investments for the modified problem lie in the convex part of the original production possibility set. This allows us to show that the safe standard for the modified problem is also a safe standard for the original problem.

Proposition 2 *If there is some $x \geq \hat{x}$ such that $f(x,a) > x$, and:*

$$\inf_{\max(f(x,a)-\xi,0) \leq z \leq x} \delta E\left[\frac{u'(f(x,r)-z)}{u'(f(x,a)-z)} f'(x,r)\right] > 1, \tag{4.3}$$

then $f(x,a)$ is a safe standard of conservation.

In the deterministic version of this model with an S-shaped production function, the condition for existence of a safe standard of conservation is that at some positive input level, the average productivity of the resource growth function should exceed the discount rate; that is, the production function should be delta-productive (see, among others, [12,7]). Note that the welfare function plays no role in this condition for existence of a safe standard. The condition given in Proposition 2 should be looked at as a modification of this delta-productivity condition in the stochastic model. The requirement is of the form $\delta E[\Psi(x,r)f'(x,r)] > 1$ where the term $\Psi(x,r)$ represents welfare effects involving ratio of marginal utilities from consumption and is directly linked to the stochastic nature of the model.

The interpretation of the condition in Proposition 2 is as follows. Given the worst production from some investment level consider a policy that depletes the resource below the original level. If all such policies have a marginal value of consumption strictly less than the expected discounted marginal value of investment, then it must be the case that the optimal investment is one that sustains the stock. Since optimal investment is monotonic in current stock, the stock is conserved under all productivity shocks and from any larger initial stock. The fact that the resource production function is stochastic implies that the marginal value of investment is evaluated over all possible realizations of the environmental disturbance. The ratio of the marginal value of investment to the marginal utility of current consumption generally differs across states of nature. The welfare effects associated with $\Psi(x, r)$ represent a lower bound on the ratio of the marginal gain in value from an increase in investment to the marginal welfare sacrificed by the corresponding reduction in current consumption. Thus, unlike the deterministic case, the welfare function plays a crucial role in determining whether or not a safe standard exists in the stochastic model.

Note that $\Psi(x, r) < 1$ for each $r > a$ so that the condition is actually stronger than requiring that delta-productivity hold in “expected terms”. This is not surprising once we consider the fact that the condition is designed to ensure that even under the worst environmental shock, the stock size is sustained from a certain level onwards. Also observe that $\Psi(x, r) = 1$ if there is no production uncertainty and in that case the condition in Proposition 2 simply reduces to the standard delta-productivity condition found in deterministic models.

4.2 Global conservation

Now, we examine the conditions that ensure global conservation. We continue to focus on the optimal policy generated by $X_m(y)$.

Recall that our definition of global conservation requires that for every $\epsilon > 0$, there exists $y^* \in (0, \epsilon)$ such that y^* is a safe standard of conservation. From the discussion in the previous subsection, it is easy to see that to ensure global conservation it is sufficient to show that there exists $\epsilon > 0$ such that

$$f(X_m(y), a) \geq y \; for \; all \; y \in (0, \epsilon). \tag{4.4}$$

In particular, this requires that $f(x, a) \geq x$ in a neighborhood of zero: even under the worst shock, the resource *production function* should not be characterized by critical depensation.

In our analysis of a safe standard of conservation, we were able to obtain considerable leverage by taking the convex hull of production possibilities and by studying the modified dynamic optimization problem on a convex feasible set. This was a fruitful approach because the best hope for finding a safe standard of conservation is in the region where average productivity of the resource is maximized. Further, for stocks above this region, and for the class of resource production functions admissible under (T.1)-(T.7), the convex hull coincides with the original production possibilities for the resource. Unfortunately, this approach is not useful in analyzing global conservation, because it requires conservation in a neighborhood of zero, which is precisely where resource production possibilities are most likely to exhibit non-convexities.

Proposition 3 *Suppose that $\nu > 1$. Then global conservation is optimal if the following condition holds:*

$$\liminf_{x \downarrow 0} \delta E \left[\frac{u'(f(x,r))}{u'(f(x,a)-x)} f'(x,r) \right] > 1. \tag{4.5}$$

Proposition 3 is the natural analogue of Proposition 2 for stocks approaching zero. The basic idea underlying condition (4.5) in Proposition 3 is straightforward: if under the worst production shock from stocks close to zero, a policy that further depletes the stock has a marginal value of consumption that is strictly less than the (expected discounted) marginal value of zero investment, then optimal investment must be one that leads to conservation.

In deterministic versions of the model, the condition for global conservation is typically a requirement that the net marginal productivity at zero (the intrinsic growth rate) exceed the discount rate. One can look at the expression:

$$\liminf_{x \downarrow 0} E \left[\frac{u'(f(x,r)}{u'(f(x,a)-x)} f'(x,r) \right] - 1.$$

as the expected welfare-modified intrinsic growth rate of the specie which has to exceed the discount rate $[\frac{1}{\delta} - 1]$ in order for global conservation to be optimal. Note that the expression above is smaller than the expected net marginal productivity at zero. Therefore, as in the case of the condition for safe standard, our condition for global conservation is stronger than "expected delta-productivity".[18]

4.3 Non-existence of safe standard and global extinction

In this final subsection, we outline the conditions under which there is no safe standard of conservation as well as conditions under which it is optimal to lead the resource towards extinction from all stocks with probability one. For this purpose we shall focus on the optimal investment function given by $X^M(y)$, so that our condition would ensure the same properties for any measurable selection from $X(y)$.

In the deterministic version of the model, global extinction occurs if the marginal productivity of investment never exceeds $(1/\delta)$. In our stochastic model, we will show that if the resource is not delta-productive from any stock in an expected sense, then a weaker result holds: there is no safe standard of conservation.

The proof of this result uses the Ramsey-Euler equation and requires that optimal consumption be positive. Observe that if for any $x > 0$, marginal utility of consuming an amount $[f(x,a) - x]$ is negative, then as noted in Proposition 1, there is always a safe standard of conservation (no matter how severe the discounting). Therefore, our condition for non-existence of safe-standard will also require that $f(x,a) - x < \xi$ (where ξ is defined in (U.3)) for all relevant x.

[18] Unlike the condition outlined for existence of safe standard, when uncertainty vanishes the condition for global conservation does not reduce to the corresponding condition in the deterministic model.

Proposition 4 *Suppose that optimal consumption is strictly positive from all positive initial stocks below $\bar{x}$ (that is, $X_M(y) < y$ for all $y \in (0, \bar{x}]$) and that $f(x, a) - x < \xi$ for all $x \in [0, \bar{x}]$. In addition, suppose that $\delta E[f'(x, r)] < 1$ for all $x \in (0, \bar{x}]$. Then, there does not exist a safe standard of conservation.*

In the previous two subsections, we have seen that the sufficient conditions for conservation in the stochastic model are stronger than the "expected" version of the conditions for conservation in comparable deterministic models. Thus, simply requiring the technology to be delta-productive in expected terms is not sufficient to ensure conservation. It is therefore intuitive that the conditions for non-existence of safe standard ought to be weaker than the "expected" version of the conditions found in deterministic model. Thus, if the resource is never delta-productive in expected terms (that is, the expected growth rate of the specie is always lower than the discount rate), then this ought to be sufficient for non-existence of safe standard.

The non-existence of a safe standard of conservation does not necessarily imply that optimal stocks converge to zero globally. To ensure global extinction, one needs stronger conditions. The next proposition outlines such conditions.

Proposition 5 *Assume that the antecedent of Proposition 4 holds. Further, assume that at least one of the following holds:*

(i) *there exists $\alpha > 0$ such that $f(x, b) < x$ for all $x \in (0, \alpha)$;*
(ii) *there exists $\widetilde{x} > 0$ such that $f(x, a) > x, \forall x \in (0, \widetilde{x})$ and there exists $\alpha > 0$ such that $f(\alpha, b) < \widetilde{x}$, and*

$$\frac{\delta E[u'(\beta(f(x, r)))f'(x, r)]}{u'(f(x, b) - x)} < 1, \forall x \in (0, \alpha),$$

where $\beta(y) \in (0, y)$ is defined by

$$f(y - \beta(y), a) = y.$$

Then, global extinction is optimal.

Note that condition (i) of Proposition 5 corresponds to a situation where the production function exhibits critical depensation even under the best realization of the random shock. Condition (ii) is a rather strong restriction on the welfare-modified expected delta-productivity and applies only to resources which do not exhibit critical depensation even under the worst realization of the random shock.

5 Proofs

Proof of Lemma 1. We show that $X_m(y)$ is non-decreasing (the proof for X^M is similar). Suppose, to the contrary, that there exist $y_1, y_2, x_1, x_2 \in Y, x_1 = X_m(y_1), x_2 = X_m(y_2)$ such that

$$y_1 < y_2, x_1 > x_2.$$

This implies that $0 \le x_2 < x_1 \le y_1 < y_2$ and that

$$y_2 - x_2 > y_1 - x_2 \ge 0, y_2 - x_1 > y_1 - x_1 \ge 0.$$

Concavity of u implies

$$u(y_2 - x_2) - u(y_2 - x_1) \le u(y_1 - x_2) - u(y_1 - x_1).$$

Since x_2 is feasible from stock y_1, x_1 is feasible from stock y_2 and $x_2 < x_1 = X_m(y_1)$, we have $x_2 \notin X(y_1)$. Then, (3.1) implies

$$\begin{aligned} u(y_1 - x_1) + \delta E[V(f(x_1, r))] &> u(y_1 - x_2) + \delta E[V(f(x_2, r))]. \\ u(y_2 - x_1) + \delta E[V(f(x_1, r))] &\le u(y_2 - x_2) + \delta E[V(f(x_2, r))]. \end{aligned}$$

Combining the above two inequalities yields:

$$\begin{aligned} &u(y_2 - x_2) - u(y_2 - x_1) \\ &\ge \delta E[V(f(x_1, r)) - V(f(x_2, r))] > u(y_1 - x_2) - u(y_1 - x_1), \end{aligned}$$

which is a contradiction.

Proof of Lemma 2. Suppose that for some $y > 0$, we have $X_m(y) = 0$. Consider an alternative policy from y, where $\varepsilon \in (0, y)$ is invested, $(y - \varepsilon)$ is consumed in the initial period, and the entire output $f(\varepsilon, r) > 0$ is consumed in the next period. From the definition of optimality, we must have:

$$\begin{aligned} 0 &\le [u(y) + \delta u(0)] - [u(y - \varepsilon) + \delta E u(f(\varepsilon, r))] \\ &= \varepsilon \left[\left\{ \frac{(u(y) - u(y - \varepsilon))}{\varepsilon} \right\} - \left\{ \frac{\delta(E u(f(\varepsilon, r)) - u(0))}{\varepsilon} \right\} \right] \\ &= \varepsilon \left[\left\{ \frac{(u(y) - u(y - \varepsilon))}{\varepsilon} \right\} - E \left\{ \frac{(u(f(\varepsilon, r)) - u(0))}{f(\varepsilon, r)} \right\} \left\{ \frac{\delta f(\varepsilon, r)}{\varepsilon} \right\} \right]. \end{aligned}$$

For ϵ near zero, the right hand expression above is negative, by using (3.3), and this contradiction establishes the result.

Proof of Lemma 3. First we establish (3.4a). Consider $y > 0$ such that $c(y) > 0$. Choose $0 < \epsilon < c(y)$. From (3.1), it follows that

$$u(c(y)) + \delta E[V(f(x(y), r))] \ge u(c(y) - \epsilon) + \delta E[V(f(x(y) + \epsilon, r))],$$

so that

$$\begin{aligned} u(c(y)) - u(c(y) - \epsilon) &\ge \delta E[V(f(x(y) + \epsilon, r)) - V(f(x(y), r))] \\ &\ge \delta E[u(\widetilde{c}(r) + \eta(\epsilon, r)) - u(\widetilde{c}(r))], \end{aligned}$$

where $\widetilde{c}(r) = c(f(x(y), r)), \eta(\epsilon, r) = f(x(y) + \epsilon, r) - f(x(y), r)$. Using concavity of u, we obtain:

$$u'(c(y) - \epsilon) \ge \delta E \left[\{u'(\widetilde{c}(r) + \eta(\epsilon, r))\} \left\{ \frac{\eta(\epsilon, r)}{\epsilon} \right\} \right].$$

Taking the $\liminf$ as $\epsilon \downarrow 0$ on both sides of the inequality (and using Fatou's lemma), we obtain (3.4a).

Next, consider the proof of (3.4b). Consider $\widehat{y} > 0$ such that $0 < c(\widehat{y}) < \widehat{y}$ and, in addition, $c(y) > 0, \forall y \in [f(x(\widehat{y}), a), f(x(\widehat{y}), b)]$. Since, (3.4a) holds, it is sufficient to show that

$$u'(c(\widehat{y})) \leq \delta E[u'(c(f(x(\widehat{y}), r)))f'(x(\widehat{y}), r)]. \tag{5.1}$$

As $c(y) > 0, \forall y \in [f(x(\widehat{y}), a), f(x(\widehat{y}), b)]$, using the upper-hemicontinuity of the correspondence $X(y)$, it can be shown that[19]

$$\vartheta = \inf\{c : c = y - x, x \in X(y), y \in [f(x(\widehat{y}), a), f(x(\widehat{y}), b)]\} > 0.$$

Let $K = \max\{\widehat{y}, \overline{x}\}$. Since f is continuous on $\Re_+ \times [a, b]$, it is uniformly continuous on $[0, K] \times [a, b]$. Therefore, if we choose $\epsilon > 0$ to be small enough, then

$$\min_{r \in [a,b]} \{f(x(\widehat{y}), r) - f(x(\widehat{y}) - \epsilon, r)\} < \frac{\vartheta}{2}.$$

In particular, let $\epsilon < \frac{x(\widehat{y})}{2}$. From (3.1), it follows that

$$u(c(y)) + \delta E[V(f(x(y), r))] \geq u(c(y) + \epsilon) + \delta E[V(f(x(y) - \epsilon, r))],$$

so that

$$\begin{aligned} u(c(y) + \epsilon) - u(c(y)) &\leq \delta E[V(f(x(y), r)) - V(f(x(y) - \epsilon, r))] \\ &\leq \delta E[u(\overline{c}(r)) - u(\overline{c}(r) - g(\epsilon, r))], \end{aligned}$$

where $\overline{c}(r) = c(f(x(\widehat{y}), r)), g(\epsilon, r) = f(x(\widehat{y}), r) - f(x(\widehat{y}) - \epsilon, r)$. Observe that, $\overline{c}(r) \leq \xi$ and $g(\epsilon, r) < \frac{\vartheta}{2} < \overline{c}(r), \forall r \in [a, b]$. Using concavity of u, we obtain:

$$u'(c(y) + \epsilon) \leq \delta \int_a^b \left[\{u'(\overline{c}(r) - g(\epsilon, r))\} \left\{ \frac{g(\epsilon, r)}{\epsilon} \right\} \right] d\mu(r). \tag{5.2}$$

Note that for a.e. $r \in [a, b], 0 \leq [\{u'(\overline{c}(r) - g(\epsilon, r))\}] < u'(\frac{\vartheta}{2})$ and further (using T.3), $\max\{f'(z, r) : \frac{x(\widehat{y})}{2} \leq z \leq x(\widehat{y}), r \in [a, b]\} < \infty$. Therefore, one can take the limit as $\epsilon \to 0$ on both sides of (5.2) and use the dominated convergence theorem to establish (5.1).

Proof of Proposition 1. Since $u'(f(x, a) - x) \leq 0$, we have $X_m(f(x, a)) \geq x$, and $f(X_m(f(x, a)), a) \geq f(x, a)$ so that (4.2) holds at $\widehat{y} = f(x, a)$.

Proof of Proposition 2. Recall the definitions of $\hat{x}(r)$ and $\hat{x}$ in Section 2, and define a modified production function $F(x, r)$ as follows:

$$F(x, r) = \begin{cases} \left[\dfrac{f(\hat{x}(r), r)}{\hat{x}(r)} \right] x \;\; for \; x \in [0, \hat{x}(r)), \\ f(x, r) \qquad\qquad for \; x \geq \hat{x}(r). \end{cases}$$

[19] See, for example, the proof of Lemma 2A, a part of the proof of Theorem 5 in [14].

Clearly, $F(x,r)$ is concave in x for all r, and F is identical to f for $x \geq \hat{x}$.

We first show that in the modified dynamic optimization problem in which F replaces f, there exists a safe standard of conservation which lies above $f(\hat{x}, a) = F(\hat{x}, a)$.

For the modified problem, let W denote the value function, and χ the optimal investment policy correspondence.[20] Define $\chi_m(y) = \min\{x : x \in \chi(y)\}$. Since $F(x,r)$ is concave in x, it is easy to show that $W(y)$ is concave in y. Denote the right hand derivative of W at any $y > 0$ by $W'_+(y)$. If c is an optimal consumption from y in the modified problem, then it can be shown that:[21]

$$W'_+(y) \geq u'(c),$$

and if $c > 0$, then W is differentiable at y, with:

$$W'(y) = u'(c).$$

Let $x \in (\hat{x}, \bar{x})$, and define $y' = F(x,a) = f(x,a)$, and $x' = \chi_m(y')$. We will now show that $x' \geq x$, so that the stock $y' = f(x,a)$ is a safe standard of conservation in the modified problem. Suppose, on the contrary that $x' < x$. Since x' is an optimal investment from stock $y' = f(x,a)$, it must be the case that $u'(f(x,a) - x') \geq 0$, so that $x' \geq \max\{f(x,a) - \xi, 0\}$. Thus, we get:

$$x > x' \geq \max\{f(x,a) - \xi, 0\}. \tag{5.3}$$

Observe now that $y' - x' = f(x,a) - x' > 0$ and therefore the principle of optimality yields for $0 < \varepsilon < y' - x'$,

$$u(y' - x') - u(y' - x' - \varepsilon) \geq \delta E[W(F(x' + \varepsilon, r)) - W(F(x', r))].$$

Using Fatou's lemma, this yields:

$$\begin{aligned}
&u'(y' - x') \\
&\geq \liminf_{\varepsilon \downarrow 0} \delta E\left\{\left[\frac{W(F(x'+\varepsilon, r)) - W(F(x', r))}{F(x'+\varepsilon, r) - F(x', r)}\right]\left[\frac{F(x'+\varepsilon, r) - F(x', r)}{\varepsilon}\right]\right\} \\
&\geq \delta E\left\{\liminf_{\varepsilon \downarrow 0}\left[\frac{W(F(x'+\varepsilon, r)) - W(F(x', r))}{F(x'+\varepsilon, r) - F(x', r)}\right]\left[\frac{F(x'+\varepsilon, r) - F(x', r)}{\varepsilon}\right]\right\} \\
&= \delta E\{W'_+(F(x', r))F'(x', r)\}.
\end{aligned} \tag{5.4}$$

Since $x' < x$, we get:

$$\begin{aligned}
&\delta E\{W'_+(F(x', r))F'(x', r)\} \geq \delta E\{W'_+(F(x, r))F'(x, r)\} \\
&\geq \delta E\{u'(F(x, r) - \chi_m(F(x, r))F'(x, r)\} \\
&\geq \delta E\{u'(F(x, r) - x')F'(x, r)\}.
\end{aligned} \tag{5.5}$$

[20] In our framework, u is not assumed to be strictly concave, and $F(x,r)$ is not strictly concave when $\hat{x} > 0$. Thus, there need not be a unique solution to the maximization problem on the right hand side of the functional equation of dynamic programming.

[21] See, for example, [16].

the last inequality in (5.5) following from the fact that $\chi_m(F(x,r)) \geq x' = \chi_m(F(x,a))$. Since $x > \hat{x}$, we have $F(x,r) = f(x,r)$ and $F'(x,r) = f'(x,r)$. Thus, (5.4) and (5.5) yield:

$$u'(f(x,a) - x') \geq \delta E\{u'(f(x,r) - x')f'(x,r)\}. \tag{5.6}$$

But this contradicts condition (4.3) of the Proposition, given (5.3).

Thus, we have established that $x' = \chi_m(f(x,a)) \geq x$. That is, in the modified dynamic optimization problem, the stochastic process of optimal investments generated by χ_m starting from any initial stock $y \geq f(x,a)$ is bounded below by x for almost every $\omega \in \Omega$, and the optimal stocks are bounded below by $f(x,a)$ for almost every $\omega \in \Omega$. Since $x > \hat{x}$, for any initial stock $y_0 \geq f(x,a)$, the policy generated by χ_m is feasible in the original (non-convex) dynamic optimization problem. As the feasible set in the modified problem always includes the feasible set in the original problem, the policy generated by χ_m from $y_0 \geq f(x,a)$ must also be optimal in the original problem.

Now, consider the optimal policy generated by X_m in the original optimization problem and suppose that $X_m(f(x,a)) < x$. The expected discounted sum of utility generated by this policy from initial stock $f(x,a)$ must be exactly identical to that generated by the policy χ_m from the same initial stock (since both are optimal in the original problem). But since χ_m is also optimal in the modified problem, the policy generated by X_m must also be optimal in the modified problem. In other words, $X_m(f(x,a)) \in \chi(f(x,a))$. Then, $\chi_m(f(x,a)) \geq x > X_m(f(x,a))$ and $\chi_m = \min\{x : x \in \chi(y)\}$ yield a contradiction.

Therefore, $X_m(f(x,a)) \geq x$ and $f(x,a)$ is always a safe standard of conservation for the original problem.

Proof of Proposition 3. First we show that $X_m(y) > 0$ for all $y > 0$. This follows directly from Lemma 2 if $\lim_{c\downarrow 0} u'(c) = +\infty$. So, consider the situation when $\lim_{c\downarrow 0} u'(c) < +\infty$. Since $f(x,r) \geq f(x,a) > f(x,a) - x$ for $x > 0$ and u is concave, (4.5) implies that $\delta E[D_+ f(0,r)] > 1$ and using Lemma 2, we have the result. Next, we show that there exists $\eta > 0$ such that $f(X_m(y),a) \geq y$ for all $y \in (0,\eta)$.

Suppose on the contrary that there exist sequences $\{x_n\}$ and $\{y_n\}$, with $x_n \downarrow 0$ and $y_n \downarrow 0$ as $n \to \infty$, such that:

$$f(x_n,a) < y_n \; and \; x_n = X_m(y_n) \; for \; n \geq 1.$$

Then, we have:

$$u'(y_n - x_n) \leq u'(f(x_n,a) - x_n) \; for \; n \geq 1. \tag{5.7}$$

Since $0 < x_n < y_n$ for $n \geq 1$, $c_n = y_n - x_n > 0$, the inequality (3.4a) yields:

$$\begin{aligned} u'(y_n - x_n) &\geq \delta E\{u'(f(x_n,r) - X_m(f(x_n,r))f'(x_n,r)\} \\ &\geq \delta E\{u'(f(x_n,r))f'(x_n,r)\}. \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we get:

$$\delta E\left[\left\{\frac{u'(f(x_n,r))}{u'(f(x_n,a) - x_n)}\right\} f'(x_n,r)\right] \leq 1. \tag{5.9}$$

Letting $n \to \infty$ in (5.9), we contradict condition (4.5) of the Proposition.

Proof of Proposition 4. Suppose to the contrary that there exists a safe standard of conservation $y^* > 0$ so that under the policy generated by the optimal investment function X^M, starting from any initial stock $y \in (y^*, \bar{x}]$, the optimal stocks $\{y_t(y, \omega)\}$ are bounded below by y^* for almost every $\omega \in \Omega$.

Let $\widetilde{x} > 0$ be defined by $f(\widetilde{x}, a) = y^*$. Then, $X^M(y) \geq \widetilde{x}$ for all $y \in (y^*, \bar{x}]$. Using upper-hemicontinuity of the correspondence $X(y)$, there exists $x \in X(y^*)$ such that $x \geq \widetilde{x}$. Therefore, $X^M(y^*) \geq \widetilde{x}$ and $X^M(y) \geq \widetilde{x} > 0$ for all $y \in [y^*, \bar{x}]$.

By assumption, $y - X^M(y) > 0$ for all $y \in (0, \bar{x}]$. Define:

$$\widehat{c} = \inf\{(y - X^M(y)) : y \in [y^*, \bar{x}]\}.$$

Using the upper hemi-continuity of the optimal investment correspondence $X(y)$, it can be shown[22] that (i) $\widehat{c} > 0$, (ii) there exists $\widehat{y} \in [y^*, \bar{x}]$ such that $\widehat{c}$ is the optimal consumption from stock $\hat{y}$; that is,

$$\hat{y} - X^M(\hat{y}) = \widehat{c}, \tag{5.10}$$

and (iii) from any initial stock $y \in [y^*, \bar{x}]$, for $t \geq 1$,

$$c_t(y, \omega) = y_t(y, \omega) - X^M(y_t(y, \omega)) \geq \widehat{c} > 0, \; for\ a.e.\ \omega \in \Omega. \tag{5.11}$$

It can verified that that $\widehat{c} < \xi$ so that $u'(\widehat{c}) > 0$.[23] Given the interiority of the optimal policy generated by X^M from $y \in [y^*, \bar{x}]$, (3.4b) yields:

$$u'(\hat{y} - X^M(\hat{y})) = \delta E\{u'(y_1(y, \omega) - X^M(y_1(y, \omega)))f'(X^M(\hat{y}), \omega_1)\}. \tag{5.12}$$

Using (5.10) and (5.11) in (5.12), we obtain: $u'(\widehat{c}) \leq u'(\widehat{c})\delta E\{f'(X^M(\hat{y}), \omega_1)\}$ which contradicts the fact that $\delta E\{f'(X^M(\hat{y}), \omega_1)\} < 1$. Thus, there is no safe standard of conservation.

Proof of Proposition 5. Proposition 4 implies that there does not exist a safe standard of conservation. We first show that this implies that for all $h > 0$:

$$\sup\left\{\frac{f(X^M(y), a)}{y} : y \geq h\right\} < 1. \tag{5.13}$$

To see (5.13), suppose on the contrary there exists some $h > 0$ for which

$$\sup\left\{\frac{f(X^M(y), a)}{y} : y \geq h\right\} \geq 1.$$

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[22] See, for example, the proof of Lemma 2A, a part of the proof of Theorem 5, in [14].

[23] Suppose not. Since optimal consumption always lies in $[0, \xi]$, it must be the case that $\widehat{c} = \xi$. As $\widehat{c} = \inf\{y - X^M(y) : y \in [y^*, \overline{x}]\}$, we have $y - X^M(y) = \xi$ and, indeed, $X(y) = \{y - \xi\}, \forall y \in [y^*, \overline{x}]$. The antecedent of Proposition 4 requires that $f(x, a) - x < \xi, \forall x \in [0, \overline{x}]$ so that $f(X^M(y), a) < X^M(y) + \xi = y - \xi + \xi = y, \forall y \in [y^*, \overline{x}]$. Note that $f(X^M(y), a) = f(y - \xi, a)$ is continuous on $[y^*, \overline{x}]$. Since $f(X^M(y^*), a) < y^*$, there exists $\epsilon > 0$ such that $f(X^M(y), a) < y^*$ for $y \in (y^*, y^* + \epsilon)$ and this contradicts the hypothesis that $\forall y > y^*$, $\{y_t(y, \omega)\}$ lies above y^* with probability one.

We will show that this implies the existence of a safe standard of conservation, a contradiction.

Let $G(y,r) = f(X^M(y),r)$. Since G is non-decreasing in y, it is easy to check that there are two possibilities: (a) there exists $y^* \geq h$ such that $G(y^*,a) \geq y^*$ and that for all $y > y^*,\ G(y,a) \geq y^*$; (b) $G(y,a) < y$ for all $y \geq h$, but there exists $y^* \geq h$ such that $\lim_{y \downarrow y^*} G(y,a) = y^*$, and for all $y > y^*$, $G(y,a) \geq y^*$.

In case (a), consider any $y \geq y^*$. We claim that for all $t \geq 0$, $y_t(y,\omega) \geq y^*$ for almost every $\omega \in \Omega$. Clearly, this is true for $t = 0$. We suppose this is true for $t = 0,\ldots,T$. Then, we have $y_{T+1}(y,\omega) = G(y_T(y,\omega),\omega_{T+1}) \geq G(y^*,a) \geq y^*$ a.e. $\omega \in \Omega$. Thus, by induction, y^* is a safe standard of conservation.

In case (b), consider any $y > y^*$. We claim that for all $t \geq 0$, $y_t(y,\omega) > y^*$ for almost every $\omega \in \Omega$. Clearly, this is true for $t = 0$. We suppose this is true for $t = 0,\ldots,T$. Then, $y_{T+1}(y,\omega) = G(y_T(y,\omega),\omega_{T+1}) > G(y_T(y,\omega),a)$ for almost every $\omega \in \Omega$. Since $y_T(y,\omega) > y^*$ by the induction hypothesis, $G(y_T,a) \geq \lim_{y \downarrow y^*} G(y,a) = y^*$ (since G is non-decreasing in y). Thus, we have $y_{T+1}(y,\omega) > y^*$ for almost every $\omega \in \Omega$. Once again, by induction, we have that y^* is a safe standard of conservation.

Thus, we have demonstrated that (5.13) holds for all $h > 0$. We now claim that if either condition (i) or condition (ii) in the statement of the proposition holds, then there exists $\epsilon > 0$ such that

$$f(X^M(y),b) < y, \forall y \in (0,\epsilon). \tag{5.14}$$

To prove this claim, note that if (i) holds, (5.14) follows immediately from the fact that

$$f(X^M(y),b) \leq f(y,b) < y, \forall y \in (0,\alpha).$$

Suppose (ii) holds. There are two possibilities: (a) $X^M(\overline{y}) = 0$ for some $\overline{y} > 0$, (b) $X^M(y) > 0, \forall y > 0$.

In case (a), since $X^M(.)$ is non-decreasing, (5.14) holds for $\epsilon = \overline{y}$.

Next, consider case (b). Since, $f(y - \beta(y),a) = y$ and (5.13) holds, $X^M(y) < y - \beta(y), \forall y \in (0,\widetilde{x})$. Let $c(y) = y - X^M(y)$. Then, $c(y) > \beta(y), \forall y \in (0,\widetilde{x})$. Suppose, contrary to (5.14) that there exists $y \in (0,\alpha)$ for which

$$f(X^M(y),b) \geq y.$$

Note that since $f(\alpha,b) < \widetilde{x}$, we have $y > c(y) > 0$ and $f(X^M(y),r) > c(f(X^M(y),r)) > 0, \forall y \in (0,\alpha), r \in [a,b]$.

Then, using the Ramsey-Euler equation (3.4b):

$$\begin{aligned} u'(f(X^M(y),b) - X^M(y)) &\leq u'(y - X^M(y)) \\ &= u'(c(y)) \\ &= \delta E[u'(c(f(X^M(y),r)))f'(X^M(y),r)] \\ &\leq \delta E[u'(\beta(f(X^M(y),r)))f'(X^M(y),r)], \end{aligned}$$

and this leads to an immediate contradiction to condition (ii). Thus, we have shown that (5.14) holds for $\epsilon = \alpha$.

Let $\overline{y}_t(y)$ be the realized path generated by the policy X^M from initial stock y on the set $\{\omega : \omega_t = b, \forall t \geq 1\}$. For any $y \in (0, \epsilon)$, (5.14) implies that the sequence $\{\overline{y}_t(y)\} \downarrow 0$. Let $\{y_t(y, \omega)\}$ be the stochastic process of stocks generated by the policy function X^M. Since, for all $t \geq 1$

$$y_t(y, \omega) \leq \overline{y}_t(y) \text{ a.s.},$$

it follows that as $t \to \infty$, $y_t(y, \omega) \to 0$ with probability one for all $y \in (0, \epsilon)$. It remains to show that as $t \to \infty$, for any $y \geq \epsilon$, $y_t(y, \omega) \to 0$ with probability one.

Choose any initial stock $y \geq \epsilon$ and let $K = \max\{y, \overline{x}\}$. It is easy to check that $\{\overline{y}_t(y)\}$ is bounded above by K. Therefore, the stochastic process $\{y_t(y, \omega)\}$ is uniformly bounded above by the constant K with probability one. Using (5.13) and assumption (T.3), there exists $\lambda > 0$, such that $f(X^M(y), a + \lambda) < y, \forall y \in [\epsilon, K]$.[24] Let $\{z_t\}$ be the deterministic sequence defined by $z_0 = K, z_t = f(X^M(z_{t-1}), a + \lambda)$. There exists positive $N < \infty$, such that $z_N < \epsilon$. Let

$$A = \{\omega : \exists T \geq 1, \omega_t \in [a, a + \lambda], T \leq t \leq T + N\}.$$

Observe that on the set A, we have $y_{T+N}(K, \omega) < \epsilon$ and since $y_t(y, \omega) \to 0$ with probability one for all $y \in (0, \epsilon)$, it follows that $y_t(y, \omega) \to 0$ a.s. on the set A. Finally, observe that since the random shocks $\{r_t\}$ are i.i.d. and $\mu\{r_t \in [a, a + \lambda]\} > 0$, we have $P(A) = 1$.

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[24] (T.3) implies the uniform continuity of $f(x, r)$ on $[0, K] \times [a, b]$.

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